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A contact problem in the theory of leaf spring bending

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Abstract

The weak joint bending (unbonded contact without friction) of the stack of slim non-uniform curved beams (leaves) with rectangular cross-sections is considered. Each leaf has one end clamped and the other free. The leaves have the same widths and different lengths (the lengths decrease upwards). The given loading is applied (upwards) to the lower leaf. This structure is the model of a leaf spring. The basic problem is to find the shapes of the leaves under bending. This problem is reduced to the problem of finding the densities of the forces of interaction between the leaves. The accurate formulation of the latter problem is propounded. The uniqueness of the solution of the problem is proved. The analytical solution is constructed in the special case of two uniform straight leaves.

Keywords: Multi-leaf spring; Weak bending; Contact problem; Analytical solution

1. Introduction

The leaf springs are widely used in the motor industry (Reimpell, 1986). They are also used in biomechanics—in different foot prostheses designs (Rudakov et al., 1997; Geil et al., 1999). The basic problem of the leaf springs theory is to find the shapes of the leaves under bending. If the shapes are known, the stresses can be calculated, and the different leaf spring optimization problems can be solved.

The simplest model of the leaf spring used to solve the basic problem is the stack of slim curved beams (leaves) with rectangular cross-sections. The leaves fit each other closely (without loading). There is no friction between the leaves. Each leaf has one end clamped and the other free (Fig. 1). All leaves have the same widths (in the direction perpendicular to the plane of Fig. 1). The given loading is applied perpendicular to the lower leaf (Fig. 1); the loading is uniform across the leaf (in the direction perpendicular to the plane of Fig. 1) but (generally speaking) it is not uniform along the leaf. The leaves undergo weak joint bending (with unbonded frictionless contact). The corresponding contact problem is not investigated enough. The present study propounds the accurate and general formulation of this problem. We start our analysis considering the model of one-leaf spring bending. In Section 3 the problem of the joint spring leaves bending is formulated. The uniqueness of the solution of the problem (in the general formulation) is

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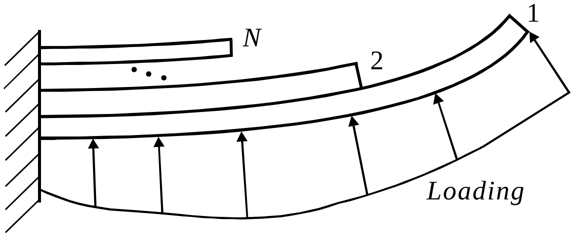

Fig. 1. The model of a multi-leaf spring (side-view).

proved in Section 4. The analytical solution of the problem in the simplest special case is constructed in Section 5. Finally we give an illustration of the solution procedure for a practical case (the calculation of the deflection of the foot prosthesis elastic element).

2. The model of one-leaf spring bending

At first consider a one-leaf spring (Fig. 2). The natural shape of the leaf is described by the function $\varphi(x)$, where x is the arc-length of the leaf segment placed between the clamped point O and some arbitrary point A; φ is the angle between the tangents to the leaf profile at these points (Fig. 2); $0 \leqslant x \leqslant \ell$; ℓ is the given overall leaf arc-length. It is assumed below that the function $\varphi(x)$ is continuous, piecewise continuously differentiable, non-decreasing, and $\varphi(x) < \pi/2$ for $0 \leqslant x \leqslant \ell$. Since $\varphi(0) = 0$, $\varphi(x) \geqslant 0$. The shape of the leaf under the load is described by the analogous function $\tilde{\varphi}(x)$. It is assumed that the potential energy stored in the leaf under bending is

$$E = \frac{1}{2} \int_0^\ell \frac{(\tilde{\varphi}'(x) - \varphi'(x))^2}{a(x)} \, \mathrm{d}x, \tag{1}$$

where the prime operator means the derivative; $a(x)$ is the given bending flexibility of the leaf, which may be expressed in terms of the Young's modulus, width and (variable) thickness of the leaf (Ziegler, 1991). The function $a(x)$ is assumed to be continuous and positive for $0 \leqslant x \leqslant \ell$. It is assumed that the bending of the leaves is weak (linear approximation with respect to the load). In this case the shape of the leaf under

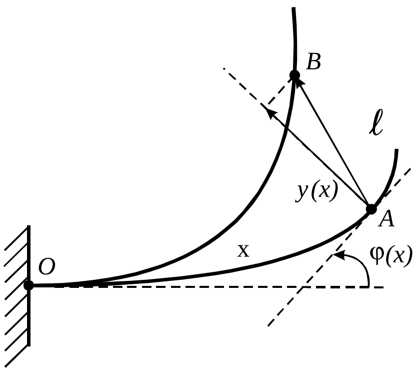

Fig. 2. The model of a one-leaf spring; the definitions of the functions $\varphi(x)$ and $y(x)$.

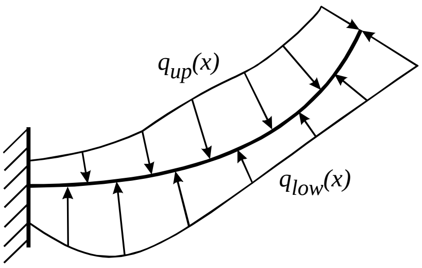

Fig. 3. The model of a one-leaf spring; the loading of the leaf.

the load may be defined by the normal displacement $y(x)$ (Fig. 2). Let B be the position of the point A of the leaf under bending. Then $y(x)$ is the projection of the vector $\overline{AB}$ onto the normal to the leaf at the point A.

It is assumed that the load is perpendicular to the leaf (Fig. 3); $q_{\text{low}}(x)$, $q_{\text{up}}(x)$ are the load densities on the lower and upper sides of the leaf. If the multiple-leaf spring is considered then this assumption should be made only for the lower side of the lowermost leaf, where the load is given a priori. For the other surfaces of the other leaves this load form follows from the absence of friction and the weakness of bending. We assume that $q_{\text{low}}(x)$, $q_{\text{up}}(x)$ can be represented in the form of

$$f(x) + \sum_i F_i \delta(x - x_i), \tag{2}$$

where $f(x) \geqslant 0$ is some piecewise continuous function, which is continuous on the right at $x = 0$ and on the left at $x > 0$; the sum is finite; $x_i > 0$; the concentrated forces $F_i \geqslant 0$; δ is the Dirac's delta function. For the lower side of the lowermost leaf, formula (2) a priori gives the form of loading. For the other surfaces of the other leaves, formula (2) is used in the following formulation of the problem.

Using (1), the principle of virtual displacements and the standard calculus of variations techniques (Lanczos, 1962), one can find that

$$y(x) = \int_0^{\ell} G(x,s)(q_{\text{low}}(s) - q_{\text{up}}(s))\,\mathrm{d}s, \tag{3}$$

where

$$G(x,s) = \int_0^{\min(x,s)} a(\tau) g(\tau,x) g(\tau,s)\,\mathrm{d}\tau \tag{4}$$

(Green's function),

$$g(\tau,x) = \int_\tau^x \cos(\varphi(x) - \varphi(\mu))\,\mathrm{d}\mu, \tag{5}$$

s, τ and μ are the variables of integrating. Integral of type $\int_b^c h(s)\delta(s - s_*)\,\mathrm{d}s$ (which may be contained in (3) and in the following analogous formulae) is considered to be equal to $h(s_*)$ in the cases $s_* = b$ or $s_* = c$.

3. The formulation of the problem of the joint spring leaves bending

Consider $N \geqslant 2$ leaves. The length of leaf k is $\ell_k > 0$ and its bending flexibility is $a_k(x)$; $1 \leqslant k \leqslant N$ (Fig. 1). The sequence ℓ_k is non-increasing one. The loading with the given density $q(x)$ (which is of type (2)) is applied to the lower side of the leaf 1. The shapes of the leaves are described by the functions $y_k(x)$ ($1 \leqslant k \leqslant N$). It is required to find these functions. In order to solve this problem, it is convenient to reformulate it so that to regard the densities $f_k(x)$ $(1 \leqslant k \leqslant N-1)$ of the forces of interaction between the leaves k, $k+1$ as the functions sought for. The functions $y_k(x)$ are expressed in terms of $f_k(x)$ as follows (these formulae result from (3) and the law of equal action and reaction)

$$y_k(x) = \int_0^{\ell_k} G_k(x,s) f_{k-1}(s)\,\mathrm{d}s - \int_0^{\ell_{k+1}} G_k(x,s) f_k(s)\,\mathrm{d}s, \tag{6}$$

where $1 \leqslant k \leqslant N$ and it should be considered that $\ell_{N+1} = 0$, $f_N(x) = 0$, $f_0(x) = q(x)$. The functions $G_k(x,s)$ are obtained by substituting $a(x) = a_k(x)$ into (4). It is assumed below that the range of index k is $1 \leqslant k \leqslant N-1$ (unless it is given explicitly). We assume that $f_k(x)$ are of type (2); it is an a priori assumption on the leaves interaction. We introduce the notation $r_k(x) = y_{k+1}(x) - y_k(x)$; then using (6) we find

$$r_k(x) = -\int_0^{\ell_k} G_k(x,s) f_{k-1}(s)\,\mathrm{d}s + \int_0^{\ell_{k+1}} (G_k(x,s) + G_{k+1}(x,s)) f_k(s)\,\mathrm{d}s - \int_0^{\ell_{k+2}} G_{k+1}(x,s) f_{k+1}(s)\,\mathrm{d}s. \tag{7}$$

The conditions of unilateral (unbonded) leaves contact may be expressed by the inequalities: $r_k(x) \geqslant 0$; besides, if $f_k(x) > 0$ then $r_k(x) = 0$. Finally we come to the following problem.

Problem. It is required to find $f_k(x)$ $(0 \leqslant x \leqslant \ell_{k+1})$, which are of type (2) and should satisfy the conditions

$$r_k(x) \begin{cases} = 0 & (f_k(x) > 0), \\ \geqslant 0 & (f_k(x) = 0), \end{cases} \tag{8}$$

where $r_k(x)$ are expressed by (7).

4. The proof of the uniqueness of the problem solution

Theorem 1. *The Problem may have only the unique solution.*

Proof. Let $f_k(x)$, $f_k^*(x)$ be the solutions of the Problem. The functions $r_k(x)$, $r_k^*(x)$ correspond to these solutions according to (7). We introduce the notation $\rho_k(x) = f_k(x) - f_k^*(x)$. Since $f_k(x)$, $f_k^*(x)$ are of type (2), $\rho_k(x)$ are also of type (2) but $f(x)$, F_i may be negative. We introduce the notation

$$\varepsilon = \sum_{k=1}^{N-1} \int_0^{\ell_{k+1}} (r_k(x) - r_k^*(x)) \rho_k(x)\,\mathrm{d}x.$$

Then it follows from (8) that $\varepsilon \leqslant 0$ (either one of the co-factors in the integral is equal to zero or these co-factors have different signs). On the other hand, using (4), (7) and introducing the notation

$$J_k(x) = \int_x^{\ell_{k+1}} g(x,s) \rho_k(s)\,\mathrm{d}s, \tag{9}$$

we obtain

$$\varepsilon = \int_0^{\ell_2} a_1(x) J_1^2(x)\,\mathrm{d}x + \sum_{k=2}^{N-1} \left[\int_0^{\ell_{k+1}} a_k(x)(J_{k-1}(x) - J_k(x))^2\,\mathrm{d}x + \int_{\ell_{k+1}}^{\ell_k} a_k(x) J_{k-1}^2(x)\,\mathrm{d}x \right] + \int_0^{\ell_N} a_N(x) J_{N-1}^2(x)\,\mathrm{d}x. \tag{10}$$

It follows from (10) that $\varepsilon \geqslant 0$. Hence, $\varepsilon = 0$. Then, if the above-mentioned properties of $\rho_k(x)$ are taken into account, it can be proved that the functions $J_k(x)$ are continuous (the simple proofs using the standard methods of mathematical analysis are not adduced). Then it follows from (10) and the equality $\varepsilon = 0$ that $J_k(x) \equiv 0$. Using (5), (9) and introducing the notation

$$H_k(x) = \int_x^{\ell_{k+1}} \cos(\varphi(s) - \varphi(x))\rho_k(s)\,\mathrm{d}s,$$

we obtain

$$J_k(x) = \int_x^{\ell_{k+1}} H_k(s)\,\mathrm{d}s. \tag{11}$$

Taking the above-mentioned properties of $\rho_k(x)$ into account, one can prove that $H_k(x)$ are the piecewise continuous functions, which are continuous on the left for $0 < x \leqslant \ell_{k+1}$ and on the right for $x = 0$. Then it follows from (11) and the equality $J_k(x) \equiv 0$ that $H_k(x) \equiv 0$, i.e. (the variable is denoted as τ)

$$\int_\tau^{\ell_{k+1}} \cos(\varphi(s) - \varphi(\tau))\rho_k(s)\,\mathrm{d}s = 0 \tag{12}$$

for $0 \leqslant \tau \leqslant \ell_{k+1}$. Multiplying (12) by $\varphi'(\tau)$ and integrating over $x \leqslant \tau \leqslant \ell_k$ yields

$$\int_x^{\ell_{k+1}} \sin(\varphi(s) - \varphi(x))\rho_k(s)\,\mathrm{d}s = 0. \tag{13}$$

It follows from (12) and (13) that

$$\int_x^{\ell_{k+1}} \cos\varphi(s)\rho_k(s)\,\mathrm{d}s = 0.$$

Using this equality and taking the above-mentioned properties of $\rho(x)$ and $\varphi(x)$ into account, one can prove that $\rho(x) \equiv 0$. Hence, $f_k(x) \equiv f_k^*(x)$. This proves Theorem 1. □

5. The analytical solution of the Problem in the simplest special case

Consider the special case: $N = 2$, $\varphi(x) \equiv 0$, $a_1(x) \equiv a_1 = \text{const.}$, $a_2(x) \equiv a_2 = \text{const.}$ (two straight uniform leaves). It follows from (4) and (5) that in this special case

$$G_k(x,s) = \begin{cases} a_k s^2(3x - s)/6 & (x \geqslant s), \\ a_k x^2(3s - x)/6 & (x \leqslant s), \end{cases} \tag{14}$$

for $k = 1, 2$. We introduce the notations

$$\Psi(\mu, s, \Lambda) = \frac{\partial}{\partial \mu}\left[\left(\frac{\mu - s}{\mu - \Lambda}\right)^3\right] = \frac{3(\mu - s)^2(s - \Lambda)}{(\mu - \Lambda)^4}, \tag{15}$$

$$\Phi(\Lambda) = \frac{1}{(\ell_2 - \Lambda)^2}\int_\Lambda^{\ell_2} (\ell_2 - s)^3 q(s)\,\mathrm{d}s + \int_\Lambda^{\ell_1} (s - \ell_2)q(s)\,\mathrm{d}s, \tag{16}$$

$$M = \int_{\ell_2}^{\ell_1} (s - \ell_2)q(s)\,\mathrm{d}s \geqslant 0. \tag{17}$$

Theorem 2. *The solution $f_1(x)$ of the Problem in the above-mentioned special case has the following form*:

(*i*) if $M = 0$ *then*

$$f_1(x) = \frac{a_1}{a_1 + a_2} q(x); \tag{18}$$

(*ii*) if $\Phi(0) \geqslant 0$ *then*

$$f_1(x) = F_1 \delta(x - \ell_2), \tag{19}$$

where

$$F_1 = \frac{3}{(a_1 + a_2)\ell_2^3} \int_0^{\ell_1} G_1(\ell_2, s) q(s)\,\mathrm{d}s; \tag{20}$$

(*iii*) if $\Phi(0) < 0$ *and* $M > 0$ *then*

$$f_1(x) = F_2 \delta(x - \ell_2) + F_3 \delta(x - \lambda) + \begin{cases} \dfrac{a_1}{a_1 + a_2} q(x) & (0 \leqslant x \leqslant \lambda), \\ 0 & (\lambda < x \leqslant \ell_2), \end{cases} \tag{21}$$

where

$$0 < \lambda < \ell_2 \textit{ is the root of the equation } \Phi(\Lambda) = 0, \tag{22}$$

$$F_2 = \frac{a_1}{(a_1 + a_2)(\ell_2 - \lambda)} \int_\lambda^{\ell_1} (s - \lambda) q(s)\,\mathrm{d}s, \tag{23}$$

$$F_3 = \frac{a_1}{(a_1 + a_2)(\ell_2 - \lambda)} \int_{\lambda+0}^{\ell_1} (\ell_2 - s) q(s)\,\mathrm{d}s. \tag{24}$$

The expression of type $b \pm 0$ denotes right-hand or left-hand limit. Note that if $q(x)$ in (21) contains the term of type $F_0\delta(x - \lambda)$ then this term is contained in $f_1(x)$ (according to non-strict inequality $x \leqslant \lambda$ in (21)).

Proof

(i) Since $q(x)$ is of type (2), $f_1(x)$ is also of type (2). Prove that conditions (8) are satisfied (here and below for $k = 1$). Substituting (18) into (7), one can find that if $\ell_1 = \ell_2$ then $r_1(x) = 0$ for $0 \leqslant x \leqslant \ell_2 = \ell_1$, and if $\ell_1 > \ell_2$ then

$$r_1(x) = -\int_{\ell_2+0}^{\ell_1} G_1(x, s) q(s)\,\mathrm{d}s. \tag{25}$$

In the latter case, using (17) and the condition $M = 0$, one can prove that $q(x) = 0$ for $\ell_2 < x \leqslant \ell_1$, hence, it follows from (25) that in this case again $r_1(x) = 0$ for $0 \leqslant x \leqslant \ell_2$. Thus, conditions (8) are satisfied in both above-mentioned cases.

(ii) It follows from (14) and (20) that $F_1 \geqslant 0$. Hence, $f_1(x)$ is of type (2). Prove that conditions (8) are satisfied. Substituting (19) into (7) and using (14)–(16) and (20), one can prove (by means of direct but somewhat cumbersome calculations) the following representation of the function $r_1(x)$:

$$r_1(x) = \frac{a_1 x^2 (\ell_2 - x)}{4\ell_2} \Phi(0) + \frac{a_1 x^3}{6} \int_x^{\ell_2} \mathrm{d}\mu \int_0^\mu \mathrm{d}s\, \Psi(\mu, s, 0) q(s). \tag{26}$$

According to (19), the inequality $f_1(x) > 0$ may be valid only at $x = \ell_2$. It follows from (26) that $r_1(\ell_2) = 0$. Besides, it follows from (26), (15) and the condition $\Phi(0) \geqslant 0$ that $r_1(x) \geqslant 0$ for $0 \leqslant x \leqslant \ell_2$. Hence, conditions (8) are satisfied.

(iii) Prove first of all that the value λ is defined correctly, i.e. the existence of the root of Eq. (22) (since the uniqueness of the solution (21) follows from Theorem 1, it is not necessary to prove the uniqueness of the root). Using (16) and (17) one can prove that $\Phi(\Lambda)$ is continuous for $0 \leqslant \Lambda < \ell_2$ and $\Phi(\ell_2 - 0) = M$. Since $\Phi(0) < 0$ and $M > 0$, the existence of the root $0 < \lambda < \ell_2$ is proved. Prove that $f_1(x)$ is of type (2). The non-strict inequality $x \leqslant \lambda$ in (21) provides the continuity on the left for $x > 0$ of that part of $f_1(x)$, which does not contain delta functions. Further, it follows from (23) that $F_2 \geqslant 0$, and it follows from (16), (22) and (24) that

$$F_3 = \frac{a_1}{(a_1 + a_2)(\ell_2 - \lambda)^3} \int_{\lambda+0}^{\ell_2} (\ell_2 - s)^3 q(s)\, \mathrm{d}s. \tag{27}$$

Hence, $F_3 \geqslant 0$ and $f_1(x)$ is of type (2). Prove that conditions (8) are satisfied. Substituting (21) into (7) and using (14), (15), (22)–(24) and (27), one can prove the following representation of the function $r_1(x)$:

$$r_1(x) = \begin{cases} 0 & (0 \leqslant x \leqslant \lambda), \\ \dfrac{a_1 (x - \lambda)^3}{6} \displaystyle\int_x^{\ell_2} \mathrm{d}\mu \int_\lambda^\mu \mathrm{d}s \Psi(\mu, s, \lambda) q(s) & (\lambda \leqslant x \leqslant \ell_2). \end{cases} \tag{28}$$

According to (21), the inequality $f_1(x) > 0$ may be valid only for $0 \leqslant x \leqslant \lambda$ and at $x = \ell_2$. It follows from (28) that $r_1(x) = 0$ for $0 \leqslant x \leqslant \lambda$ and $r_1(\ell_2) = 0$. Besides, it follows from (28) and (15) that $r_1(x) \geqslant 0$ for $\lambda \leqslant x \leqslant \ell_2$. Hence, conditions (8) are satisfied.

Note: If $q(x) = F_0 \delta(x - \ell_2)$ then the conditions (i) and (ii) of Theorem 2 are satisfied simultaneously. □

6. An illustration of the solution procedure for a practical case

If a spring with two straight uniform leaves is used as the elastic element of the foot prosthesis then one of the practical problems is to calculate the deflection d of this elastic element under the given loading (Rudakov et al., 1997). The results of Section 5 allow one to construct the solution of this practical problem. In terms of Section 2,

$$d = y_1(\ell_1). \tag{29}$$

Suppose the loading is uniform:

$$q(x) \equiv q_0 = \text{const.} \tag{30}$$

Substituting (30) into the formulation of Theorem 2 and introducing the notation

$$\alpha = \ell_2/\ell_1,$$

we obtain:

(i) if $\alpha = 1$ then

$$f_1(x) = \frac{a_1 q_0}{a_1 + a_2}; \tag{31}$$

(ii) if $0 < \alpha \leqslant 2 - \sqrt{2}$ then

$$f_1(x) = \frac{a_1 q_0 \ell_1 (\alpha^2 - 4\alpha + 6)}{8(a_1 + a_2)\alpha} \delta(x - \ell_2); \tag{32}$$

(iii) if $2 - \sqrt{2} < \alpha < 1$ then

$$f_1(x) = \frac{a_1 q_0 \ell_1 (1 - \alpha)}{2\sqrt{2}(a_1 + a_2)} [(3 + 2\sqrt{2})\delta(x - \ell_2) + \delta(x - \lambda)] + \begin{cases} \dfrac{a_1 q_0}{a_1 + a_2} & (0 \leqslant x \leqslant \lambda), \\ 0 & (\lambda < x \leqslant \ell_2), \end{cases} \tag{33}$$

where $\lambda = (1 + \sqrt{2})\ell_2 - \sqrt{2}\ell_1$.

Then it follows from (6), (29) and (31)–(33) that

$$d = \frac{a_1 q_0 \ell_1^4}{8}\left\{1 - \frac{a_1}{6(a_1 + a_2)}[\alpha(3-\alpha)(\alpha^2 - 4\alpha + 6) - \beta]\right\},$$

where

$$\beta = \begin{cases} 0 & (0 < \alpha \leqslant 2 - \sqrt{2}), \\ (\alpha - 2 + \sqrt{2})^2(1-\alpha)[8 + 6\sqrt{2} - (5 + 4\sqrt{2})\alpha] & (2 - \sqrt{2} \leqslant \alpha \leqslant 1). \end{cases}$$

7. Conclusions

The propounded approach to the investigation of the interaction of spring leaves under joint bending allows one to understand the bending of two straight uniform leaves in full. It is possible to hope that this approach will lead to determining the interaction pattern of the leaves in more complicated cases.

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